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1/*f* Noise in Systems Showing Stochastic Resonance

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Stochastic resonator systems with input and/or output 1/*f* noise have been studied. Disordered magnets/dielectrics serve as examples for the case of output 1/*f* noise with white noise (thermal excitation) at the input of the resonators. Due to the fluctuation-dissipation theorem, the output noise is related to the out-of-phase component of the periodic peak of the output spectrum. Spin glasses and ferromagnets serve as interesting examples of coupled stochastic resonators. A proper coupling can lead to an extremely large signal-to-noise ratio. As a model system, a 1/*f*-noise-driven Schmitt trigger has been investigated experimentally to study stochastic resonance with input 1/*f* noise. Under proper conditions, we have found several new nonlinearity effects, such as peaks at even harmonics, holes at even harmonics, and 1/*f* noise also in the output spectrum.

KEY WORDS: Stochastic resonance; 1/*f* noise; signal-to-noise ratio.

1. INTRODUCTION

A physical system of randomly excited independent particles in a periodically modulated double potential well shows the well-known statistical phenomenon called stochastic resonance (SR), which was first introduced as a possible explanation of the observed periodicity in the recurrences of the earth's ice ages.[1-3] According to this view, net particle flows between the wells are induced by the external modulation so that

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there exists a time-dependent population in each well which is coherent with the external field.

SR has recently become the object of many investigations stimulated primarily by an interesting laser experiment in which it was observed and quantitatively studied.[4,5] A number of papers on the theory[6–8] as well as on analog simulations[3,9,10] followed. The most important qualitative predictions of the theory and the results of the experiments/simulations can be summarized as follows:

(i) The coherent part of output signal increases from zero with increasing noise, passing through a maximum at a strength of noise excitation such that the Kramers time becomes roughly comparable to twice the period of the modulating field.[4–10]

(ii) The power spectrum of the output signal contains very sharp, strong peaks (theoretically, delta functions) located at only the odd multiples of the modulating frequency, provided the unperturbed wells are symmetric.[8,10]

(iii) Destruction of the well symmetry by the application of an external dc field (in addition to the modulation) results in the appearance of peaks at the even multiples of the modulating frequency.

Note that under special conditions, both nonlinearity predictions (ii) and (iii) are violated, as is shown in Section 3 of the present paper. Namely, in the case of symmetric wells and $1/f$ driving noise, peaks can be found also at even harmonics, and in special cases these peaks turn into holes. The last effect can be found even in the case of white driving noise, and it has been observed under certain conditions by other researchers, too.[11,12]

In practical physical systems, at not too low frequencies (say, $f > 1$ MHz), spontaneous excitations very often can be represented by a white noise (that is, by a noise with a frequency-independent power density spectrum).[13] As a consequence, most of the above-cited publications are about the case of white noise excitation. Moreover, the case of colored noise is a complicated theoretical topic.[7,8] On the other hand, in the ultra-low-frequency limit (say, $f \ll 1$ Mhz), many physical and biological systems show $1/f$ noise, which is a colored noise with a logarithmically decaying autocorrelation function.[14,15] Regarding the frequency domain, $1/f$ noise is a spontaneous fluctuation with a power density spectrum roughly proportional to $1/f$ through many decades of frequency, usually down to the frequency limit given by the finite duration of the measurement. The wide occurrence of $1/f$ noise in nature implies the problem of $1/f$ noise at SR. As $1/f$ noise has been found also in neuron signals, this problem has become more important: according to recent interesting speculations,[16–18] SR may enhance the flow of information in sensory neurons of living systems.

In this paper, we shall deal with the presence of input/output $1/f$ noise at SR. In Section 2 disordered magnets/dielectrics are viewed as stochastic resonator systems where (without ac excitation) an input white noise and an output $1/f$ noise are present. Important consequences are pointed out, such as: ac excitation, the out-of-phase component of the periodic response is connected with the background noise of SR; a ferromagnetic-like coupling between the stochastic resonators leads to a model of ferromagnets, that is, to a system with superstrong SR; and coupling of randomly mixed ferromagnetic/antiferromagnetic types leads to a model of spin glasses implying long-time relaxation and memory effects.

2. STOCHASTIC RESONANCE AND 1/f NOISE IN MAGNETS AND DIELECTRICS

First, we would like to point out that an elementary magnetization moment (spin) or an electrical polarization moment (dipole moment) in a double-well potential can be a very natural representation of SR[19] provided that the spin or dipole moment is coupled to a heat bath. A flip of spin or dipole moment can be considered as a jump from one well into another or back. An ensemble of such systems can often be a representation of random magnets/dielectrics.[20–22]

The complex ac susceptibility $\kappa(x)=\kappa'(\omega)+i\kappa''(\omega)$ of the system describes the response of magnetization/polarization of the material to an external periodic excitation. For the case of symmetric double wells, the result is well known (see refs. 21 and 22 and references therein):

$$\kappa'(\omega, T, U)=\frac{NAM}{kT[1+\omega^2\tau_0^2\exp(2U/kT)]} \tag{1}$$

$$\kappa''(\omega, T, U)=\frac{NAM\omega\tau_0\exp(U/kT)}{kT[1+\omega^2\tau_0^2\exp(2U/kT)]} \tag{2}$$

$$|\kappa(\omega, T, U)|^2=\frac{(NAM)^2}{(kT)^2[1+\omega^2\tau_0^2\exp(2U/kT)]} \tag{3}$$

where N is the number of independent double wells, A is a linear coupling term between the external field and the asymmetry induced in the depth of double wells, M is the value of the elementary magnetic polarization moment, U is the potential barrier between the wells, and τ_0 is the reciprocal mean attempt frequency.

We note that while Eqs. (1)–(3) were well known half a century ago and Eq. (3) describes the strength of the harmonic peak in the power spectrum of the magnetic/dielectric response, it describes the familiar SR, too. To show this, we have to replace the elementary thermal energy kT by the variance D of the white noise excitation in a stochastic resonator.

The analogy with SR is even more explicit if we consider the signal-to-noise ratio of the response. According to the fluctuation-dissipation theorem, the power spectral density S of thermal noise in magnetization/polarization is just

$$S(\omega, T, U) = \frac{4kT\kappa''(\omega, T, U)}{\omega} = \frac{4NAM\omega\tau_0 \exp(U/kT)}{1 + \omega^2\tau_0^2 \exp(2U/kT)]} \tag{4}$$

So, the signal-to-noise ratio (SNR) at an external periodic excitation $H_{\text{ext}} \sin(\omega t)$ is

$$\text{SNR} = \delta(\omega) \frac{H_{\text{ext}}^2 |\kappa(\omega, T, U)|^2}{S(\omega, T, U)} = \delta(\omega) \frac{H_{\text{ext}}^2 NAM}{4(kT)^2 \tau_0 \exp(U/kT)} \tag{5}$$

which, apart from some well parameters, is exactly the formula previously obtained for SR in the adiabatic, small-perturbation approximations.(4–6) This perfect analogy is not surprising, because the fluctuation of the thermal energy on the thermal energy on the microscopic scale (κt) is indeed known to be a white noise well beyond microwave frequencies at not too low temperatures.(13)

In disordered magnets/dielectrics there is a wide distribution $g(U)$ of barrier energies.(21,22) In this case, the resultant susceptibility components κ'_{res} and κ''_{res} are given naturally from the integrals

$$\kappa'_{\text{res}}(\omega, T) = \int \kappa'(\omega, T, U)\, g(U)\, dU \tag{6}$$

and

$$\kappa''_{\text{res}}(\omega, T) = \int \kappa''(\omega, T, U)\, g(U)\, dU \tag{7}$$

respectively. As is well known, a uniform distribution $g(U) = \text{const}$ implies a lossy part κ''_{res} which is independent of the frequency. Then, according to the fluctuation-dissipation theorem [first part of Eq. (4)] there is a pure $1/f$ noise in the spontaneous magnetization/polarization. In practical cases, $g(U)$ is not uniform, but it is a flat distribution with a small relative change through an energy interval κt. That implies a "practical" $1/f$ noise, that is, $S(f) \propto 1/f$ through many decades of frequency.(23 25)

It is important to note that (within the inaccuracy of experiments) noise in spontaneous magnetization satisfies the first part of Eq. (4) even in spin glasses,[26] which can be viewed as systems of randomly coupled stochastic resonators (see below). That fact is very interesting, because spin-glass systems are out of thermal equilibrium, so the fluctuation-dissipation theorem does not hold there.[25,26] Moreover, $\kappa''_{res}(\omega, T)$ has a very weak frequency dependence below the melting temperature of the glassy state. Correspondingly, the spectrum of spontaneous magnetization is of $1/f$ shape even in spin glasses.[26]

In conclusion, during the measurement of ac susceptibility, disordered magnets/dielectrics can very often be represented by a system of a large number of *independent* stochastic resonators. These elementary resonators are driven by independent white noise sources (thermal excitation). A flat distribution of barrier energies yields a $1/f$ noise at the output, that is, in the resultant magnetization/polarization of these materials. Interestingly, this is the case even in spin-glass systems, where there is a strong coupling between stochastic resonators.

Before closing this section, disregarding the problem of $1/f$ noise, we would like to summarize the important points which we can learn about stochastic resonance from the example of magnetic materials and present some implications and speculations:

(a) The output background noise spectrum is related to the out-of-phase component $\kappa''(\omega)$ of the susceptibility which is given by the cosinusoidal component of the output periodic peak when a sinusoidal input periodic signal is applied. In the case of materials, $\kappa''(\omega)$ is proportional to the dissipative loss during sinusoidal excitation, that is, a small loss implies a low noise. Note that the measurement of $\kappa''(\omega)$ gives the same information as the measurement of the output background noise spectrum; however, the time requirement of the measurement (for a given accuracy) can be very much shorter when the measurement of the susceptibility is applied.

(b) The signal-to-noise ratio (SNR) is proportional to $|\kappa(\omega)|^2/\kappa''(\omega)$, that is, a large susceptibility with a small lossy part is the necessary condition to get a good result when SR is used in noise filtering applications.

(c) At this point, the analogy with magnetic materials leads us to the case of ferromagnetic materials. In technology, both the above conditions are standard requirements and high tech materials can reach $|\kappa(\omega)|^2 \approx 10^{12}$ with a very small lossy part. Consequently, a system of stochastic resonators with a properly chosen *ferromagnetic coupling* between the resonators should be the object of investigations when the aim is a large SNR. In Fig. 1, a system of detectors and coupled resonators

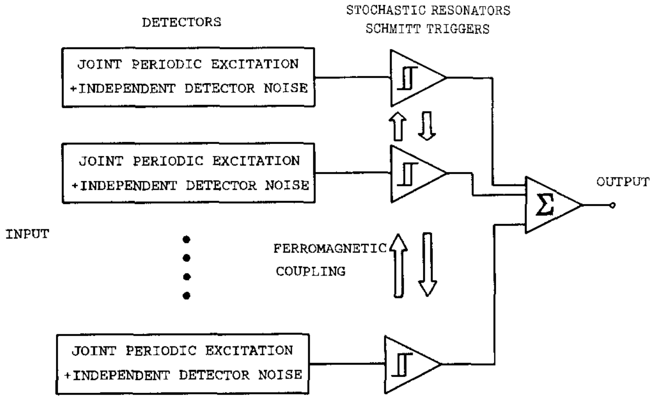

Fig. 1. Detector–stochastic resonator system to reach extra high signal-to-noise ratio (reduction of detector noise).

(*ferromagnetic coupling*) is shown which can model ferromagnets. The detectors observe the same periodic signal and add their independent detector noise to this signal. The resonators are interacting via an optimized ferromagnetic coupling and the output signal of the system is the sum of resonator outputs. This system can show a "super" stochastic resonance if its parameters are properly chosen.

(d) If in the above system (Fig. 1) the sign and strength of interaction between resonators is spatially random, then we have a model of spin glasses. It is interesting to note that while the SNR in a spin glass is much lower than in a ferromagnet, analysis of spin-glass systems may be relevant in studying information processing in the brain.[27] That arrangement may be a special application of coupled stochastic resonators for modeling biological information processing.[18]

Finally, we would like to cite some interesting articles[28–32] which give relevant approaches to the problem of coupled resonators.

3. STOCHASTIC RESONANCE WITH 1/*f* NOISE AT THE INPUT

These investigations have been carried out on a Schmitt trigger of a symmetric hysteresis with threshold voltage $+U_t$ and $-U_t$ (see Fig. 2), which represent a square-shaped double potential well with a barrier height of U_t. The input was fed by the sum of the sinusoidal signal (283.5 Hz) and a Gaussian $1/f$ noise. The key element of the noise generator was a MOS field effect transistor and the resulting noise was a rather good $1/f$ noise (power exponent: 0.96) with cutoff frequencies of 0.1 Hz and 10 kHz,

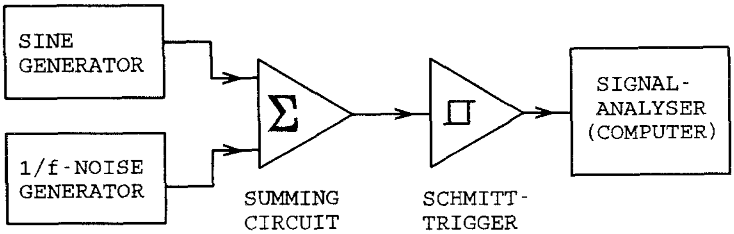

Fig. 2. Experimental setup for studying stochastic resonance with $1/f$ noise excitation.

respectively. The two-stage noise resulting at the output of the Schmitt trigger has been developed by a PC and an FFT program. The time duration of the measurement was approximately 0.1 sec. After calculating the spectrum of the output noise, the measurement sequence was repeated several hundred times and averaged to get a sufficient accuracy of results. The input and output signals of the Schmitt trigger were visualized on an oscilloscope to get information about the temporal behavior of signals and to avoid possible artifacts due to parasite effects in the electronic circuitry.

Figure 3 shows a typical experimental SR curve. Comparing the measured curve with the theoretical prediction for white noise, one can see that the resonance is less sharp in the case of $1/f$ noise. This difference originates from the low-frequency cutoff of the $1/f$ noise (0.1 Hz), which is two decades lower than the characteristic frequency (10 Hz) coming from the duration (0.1 sec) of one measurement record. In this case, the $1/f$ noise

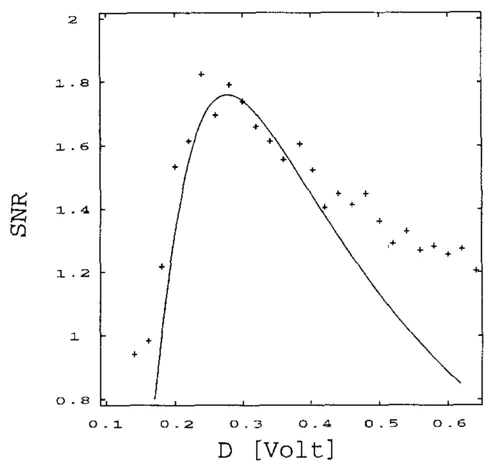

Fig. 3. Signal-to-noise ratio at the output of the Schmitt trigger at $1/f$ noise excitation, $U_t = 0.31$ V, $U_0 = 0.1$ V. Solid line represents the theoretical curve with the same U_t, U_0.

components in the range 0.1–10 Hz correspond to a slowly fluctuating asymmetry in the depth of the potential wells. This means that the instantaneous barrier height seen by the particle is slowly fluctuating. As a consequence, the optimal noise intensity for the SR maximum is also slowly fluctuating. That implies a less sharp SR for $1/f$ noise than for white noise. To prove the correctness of the above argument, we made a simulation for such a case, when the low-frequency cutoff of $1/f$ noise was identical with the reciprocal measurement time. In that case, the above effect does not exist and the SR curve for $1/f$ noise fits very well the theoretical curve for white noise (see Fig. 4).

The above-mentioned $1/f$-noise-induced (fluctuating) asymmetry leads to another interesting property of SR, namely, to peaks of the output noise spectrum at even harmonics (see Fig. 5).

When the noise is still small but the sinusoidal excitation becomes so large that U_0 is comparable to U_t, then a new interesting nonlinear phenomenon occurs: instead of peaks at even harmonics, we can observe local minima ("holes") at these frequencies (see Figs. 6 and 7). That strange behavior has been observed by others[11,12] at white noise excitation. One of the authors (Z.G.) found the explanation of this effect: when these conditions are present, the output signal of the Schmitt trigger is an almost periodic square wave. The switching time instants t_n ($n = 1, 2, \ldots$) of the square wave can be given as $t_n = Nt_0/2 + r_n$, where T_0 is the reciprocal frequency of sinusoidal excitation and r_n is a small random perturbation term. For white noise excitation, r_n is uncorrelated for any n; for $1/f$ noise excitation, r_n has a slowly decaying correlation function. For the uncorrelated case,

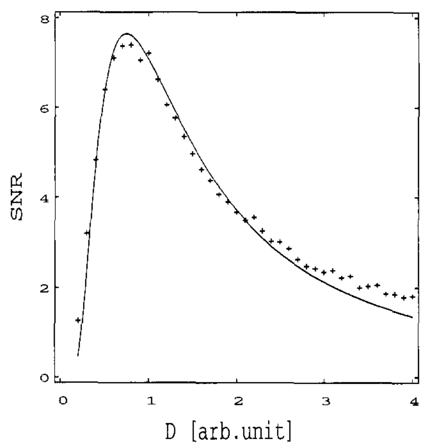

Fig. 4. Signal-to-noise ratio at $1/f$ noise driving when the low-frequency cutoff of $1/f$ noise is equal to the reciprocal time duration of the measurement record (computer modeling).

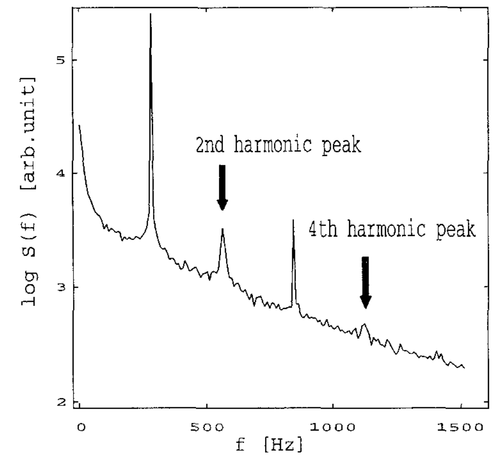

Fig. 5. Peaks in the output spectrum at even harmonics at $1/f$ noise driving (see text). $U_t = 0.31$ V, $U_0 = 0.24$ V, $D = 0.26$ V.

the power spectrum of such a perturbed square wave has zero values at even harmonics, as can be shown by elementary calculations. For the case of $1/f$ noise, the spectrum has local minima at even harmonics. Increasing the noise amplitude at the input will destroy the periodic-like behavior of the output square wave, so that several random switching events can happen within $T_0/2$, which will "wash out" the holes from the spectrum (see Figs. 6 and 7).

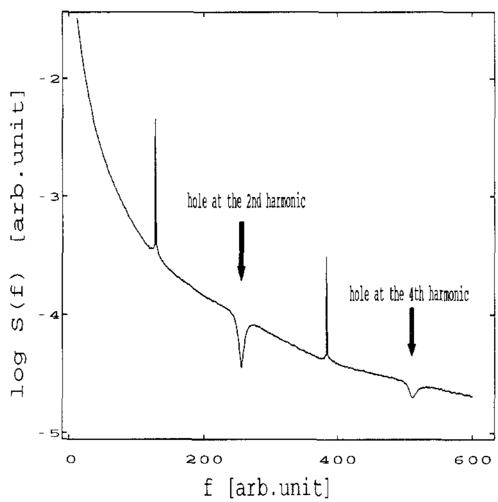

Fig. 6. Holes in the output spectrum at even harmonics at $1/f$ noise driving (see text). $U_t = 1.0$ V, $U_0 = 0.8$ V, $D = 0.2$ V.

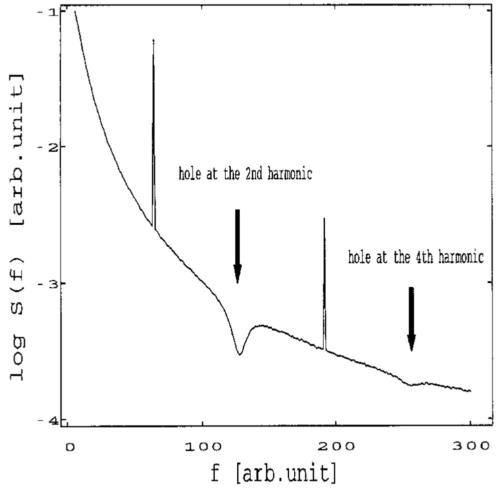

Fig. 7. Holes in the output spectrum at even harmonics at white noise driving (see text). $U_t = 1.0$ V, $U_0 = 0.8$ V, $D = 0.2$ V.

Finally, another interesting nonlinear phenomenon can be found without input periodic signal when the input $1/f$ noise is very large: $D \gg U_t$. In this case, the output spectrum turns out to be $1/f$, too (Fig. 8). That is a very remarkable effect, because under these conditions the output noise is given by the signum operation on the input noise (the hysteresis is much smaller than the driving noise amplitude). It seems that the zero-crossing-time distribution has a crucial role in determining the spectrum of

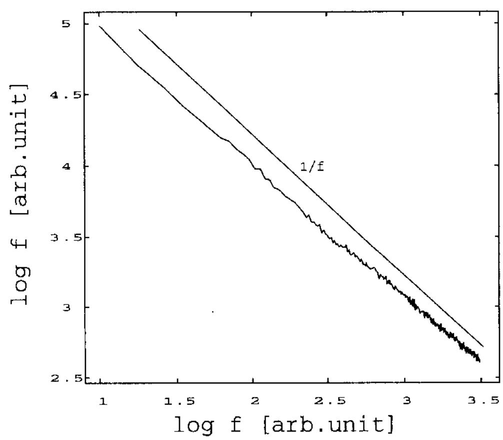

Fig. 8. Output spectrum of the Schmitt trigger at very large $1/f$ noise driving (see text), $U_t \ll D$, $U_0 = 0$ (computer modeling).

Gaussian $1/f$ noise. We note that according to further studies[33] (which are beyond the scope of the present paper), a truncation of a Gaussian $1/f$-noise time signal at arbitrary level of amplitude does not change the $1/f$ spectrum. That proves the great role of the distribution of level crossing times in the existence of the $1/f$ noise.

In conclusion, we list the remarkable new effects which have been found at $1/f$ input noise:

(a) In the linear case ($U_0 \ll D \ll U_t$), there usually is a less sharp SR curve than that at white noise input.

(b) In the case of weak nonlinearity ($U_0 \simeq D \ll U_t$) there are peaks in the output noise spectrum at even harmonics even if the double potential well is symmetric.

(c) In the case of strong nonlinearity ($D \ll U_0 \simeq U_t$), there are holes in the output noise spectrum at even harmonics.

(d) In the case of noisxe overloading the system ($U_0 \simeq 0$, $U_t \ll D$), the output noise spectrum becomes $1/f$, too. The spectrum of Gaussian $1/f$ noise turns out to be invariant against the signum operation.

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